

Cayley structures on S^6 as a twistor bundle sections

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The nearly Kähler structures on the 6-sphere, as a twistor bundle sections are researched. We show that for any point of twistor bundle there exists an 1-parametric family of sections, passing through the point, which give nearly Kähler structures on the round sphere. Some properties of those sections are found.

Keywords: Cayley structures, nearly Kähler structure, twistor bundle.

Introduction

We will talk about Cayley structures on the 6-sphere. With round metric they define nearly Kähler structures on S^6 .

Definition 1.1. *Nearly Kähler manifold is an almost Hermitian manifold (M, g, J, ω) with the property that $(\nabla_X J)X = 0$ for all tangent vectors X , where ∇ denotes the Levi-Civita connection of g . If $\nabla_X J \neq 0$ for any non-zero vector field X , (M, g, J, ω) is called strictly nearly Kähler.*

Nearly Kähler geometry comes from the concept of weak holonomy introduced by A. Gray in 1971 [1], this geometry corresponds to weak holonomy $U(n)$. Lie group $U(n)$ is a structure group of almost Hermitian manifold, if holonomy group is equal to $U(n)$ too, then this manifold is Kähler. In weak case [1] almost Hermitian manifold with weak holonomy group $U(n)$ is nearly Kähler. The class of nearly Kähler manifolds appears naturally as one of the sixteen classes of almost Hermitian manifolds described by the Gray-Hervella classification [2].

At 2002 Nagy P.-A. has proved [3] that every compact simply connected nearly Kähler manifold M is isometric to a Riemannian product $M_1 \times \dots \times M_k$, such that for each i , M_i is a nearly Kähler manifold belonging to the following list: Kähler manifolds, naturally reductive 3-symmetric spaces, twistor spaces over compact quaternion-Kähler manifolds with positive scalar curvature, and nearly Kähler 6-manifold. This is one of the reasons of interest to the nearly Kähler 6-manifolds.

For 6-dimensional nearly Kähler Riemannian homogeneous manifolds we have the following classification:

Theorem (J.P. Butruille [4]) *Nearly Kähler, 6-dimensional, Riemannian homogeneous manifolds are isomorphic to a finite quotient of G/H where the groups G, H are given in the list:*

- $G = SU(2) \times SU(2)$ and $H = \{1\}$

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- $G = Sp(2)$ and $H = SU(2)U(1)$. Then, $G/H \simeq \mathbb{C}P^3$, the 3-dimensional complex projective space.
- $G = SU(3)$, $H = U(1) \times U(1)$ and G/H is the space of flags of \mathbb{F}^3 .
- $G = G_2$ and $H = SU(3)$. In this case G/H is the round 6-sphere.

And we don't know an example of the nonhomogeneous nearly Kähler manifold.

For the first three manifolds from the list above there exists only one nearly Kähler structure. But in case of the round 6-sphere one has an infinite number of those structures.

Theorem (M. Verbitsky [5]) *Let (M^6, I, g) be a nearly Kähler manifold. Then the almost complex structure is uniquely determined by the Riemannian structure, unless M is locally isometric to a 6-sphere.*

All almost complex structures, which gives the nearly Kähler structures together with the round metric are Cayley structures on S^6 .

1 Cayley structures on S^6

Let's look \mathbb{R}^7 as the space of purely imaginary Cayley numbers with inner product

$$\langle x, y \rangle = -\text{Re}(xy), \quad \forall x, y \in \mathbb{R}^7$$

Then, sphere S^6 is a subset of unit length numbers in \mathbb{R}^7 . Inner product \langle, \rangle induces round metric g_0 on S^6 .

Let $R_x : \mathbb{O} \rightarrow \mathbb{O}$ denotes the right Cayley multiplication by x , $R_x y = y \cdot x$. The right multiplication defines standard Cayley structure \mathcal{J} on the S^6 :

$$\mathcal{J} = \{(x, R_x) : x \in S^6\}$$

For any vector $y \in T_x S^6$, $\mathcal{J}_x(y) = R_x(y) = y \cdot x$, $\forall x \in S^6$. By the multiplication properties [6], one can show that $\mathcal{J}_x(y) \in T_x S^6$ and

$$\mathcal{J}_x^2(y) = R_x(R_x(y)) = (y \cdot x) \cdot x = y \cdot (x \cdot x) = -y$$

So, the Cayley structure is an almost complex structure on S^6 .

Another Cayley structures are defined from the following way. Let $A \in O(7)$ is any orthogonal transformation, then [6] one can define structure \mathcal{J}^A :

$$\mathcal{J}^A = \{(x, A^{-1}R_{A(x)}A) : x \in S^6\},$$

Obviously, $\mathcal{J}_x^A(y) = A^{-1}R_{A(x)}A(y) = A^{-1}(A(y) \cdot A(x))$, $\forall x \in S^6$, $\forall y \in T_x S^6$.

Structure \mathcal{J}^A is equal to standard one if and only if A is automorphism of Cayley algebra. So, the space of all Cayley structures is homogeneous space $O(7)/G_2$.

Remark 1. *Further we will speak about almost complex structures which define the standard orientation, induced from \mathbb{R}^7 , only. In this case the space of those structures is $SO(7)/G_2 \cong \mathbb{R}P^7$.*

All these structures are nearly Kähler on the homogeneous Riemannian manifold $S^6 = G_2/SU(3)$. Every embedding G_2 in $SO(7)$ defines on the sphere some nearly Kähler structure associated with round metric.

2 Twistor bundle over S^6

Let $(Z^+(S^6, g_0), \pi)$ is a twistor bundle over S^6 :

$$Z^+(S^6, g_0) = \{(x, J_x) : x \in S^6, J_x : T_x S^6 \longrightarrow T_x S^6, J_x^2 = -1$$

and define the standard orientation, induced from $\mathbb{R}^7\}$

$$\pi : Z^+(S^6, g_0) \longrightarrow S^6, \pi(x, J_x) = x$$

Each fiber of bundle is 6-dimensional and diffeomorphic to $SO(6)/U(3) \cong \mathbb{C}P^3$. Thus, the space $Z^+(S^6, g_0)$ is 12-dimensional. Any almost complex structure \mathcal{J} on (S^6, g_0) is a smooth section of the bundle and $\mathcal{J}(S^6)$ is smooth submanifold in $Z^+(S^6, g_0)$.

Lemma 1. *For any point $(p, J_p) \in Z^+(S^6, g_0)$ there exists Cayley structure \mathcal{J}^A , with $\mathcal{J}_p^A = J_p$.*

Proof. Let $p \in S^6$ is any point on the sphere, and $J_p \in \pi^{-1}(p)$ is arbitrary. Let \mathcal{J}_p is value of standard Cayley structure at p . Then one can find corresponding orthogonal transformation $A \in SO(6)$ of $T_p S^6$ with $\mathcal{J}_p = A J_p A^{-1}$. Look A , as rotation around the point p in \mathbb{R}^7 . Then A defines Cayley strucutre

$$\mathcal{J}^A = \{(x, A^{-1} \mathcal{J}_{A(x)} A) : x \in S^6\}$$

At point p we have $\mathcal{J}_p^A(y) = A^{-1} \mathcal{J}_{A(p)} A(y) = A^{-1} \mathcal{J}_p A(y) = J_p(y)$ for any vector $y \in T_p S^6$. Lemma is proved.

Dimension of the fiber is equal to 6, but dimension of the Cayley structures space $SO(7)/G_2$ is equal to 7. So it would be natural to suggest that in any fiber we have points where more then one Cayley structure pass through.

Theorem 1. *For any point $(p, J_p) \in Z^+(S^6, g_0)$ there exists an 1-parametric family of twistor bundle sections, passing through the point, which give nearly Kähler structures on (S^6, g_0) .*

Proof. Let $(p, J_p) \in Z^+(S^6, g_0)$ is any point of twistor bundle over S^6 . Let \mathcal{J}^A is corresponding to J_p (as in Lemma 1), $\mathcal{J}_p^A = J_p$, $A \in SO(6)$. Group $SO(6)$ in this case is a subgroup in $SO(7)$ of rotations around the p . Let $\lambda = (\cos \varphi + i \sin \varphi) E \in U(3)$, $\varphi \in [0; 2\pi/3)$ is unitary transformation in the plane $T_p S^6$, commuting with \mathcal{J}_p . This transformation $\lambda \notin SU(3) = G_2 \cap SO(6)$, while $\varphi \neq 0$, then

$$\mathcal{J}_p^{\lambda A} = A^{-1} \lambda^{-1} \mathcal{J}_p \lambda A = A^{-1} \mathcal{J}_p A = \mathcal{J}_p^A = J_p,$$

but $\mathcal{J}^{\lambda_1 A} \neq \mathcal{J}^{\lambda_2 A}$ for any different φ_1 and φ_2 , as in this case $\lambda_1 \lambda_2^{-1} \notin G_2$. Theorem is proved.

Remark 2. *Denote 1-parametric family of Cayley structures through the point $(p, J_p) \in Z^+(S^6, g_0)$ as $\{p, J_p\}$.*

Lemma 2. *Let $(p, J_p) \in Z^+(S^6, g_0)$ then, any two Cayley structures \mathcal{J}^A and $\mathcal{J}^B \in \{p, J_p\}$ have exactly two intersection points.*

Proof. Let \mathcal{J}^A and $\mathcal{J}^B \in \{p, J_p\}$. By lemma 2 and proof of theorem 1 one can think about transformations A and B as a rotations in \mathbb{R}^7 with respect to p . Moreover $B = \lambda A$,

where $\lambda = (\cos \varphi + i \sin \varphi)E \in U(3)$, $\varphi \in (0; 2\pi/3)$. Rotation around the point p in \mathbb{R}^7 keeps point p and $-p$ invariant. Obviously, that values of structures \mathcal{J}^A and \mathcal{J}^B at $-p$ are equal.

Suppose that one can find a point $q \neq \pm p$ on the sphere, and point $J_q \in \pi^{-1}(q)$, with $\mathcal{J}_q^A = \mathcal{J}_q^{A\lambda} = J_q$. Then by definition of Cayley structure and proof of theorem 1:

$$(\lambda A)^{-1} R_{\lambda A(q)} (\lambda A)(y) = A^{-1} R_{A(q)} A(y)$$

for all $y \in T_q S^6$.

$$\lambda^{-1} R_{\lambda q'} (\lambda y') = R_{q'}(y')$$

where $q' = Aq$, $y' = Ay$. As $q' \neq \pm p, 0$ and y' is arbitrary tangent vector at point q' , then the equality is possible just if $\lambda \in G_2$, this contradicts with choice of λ .

Let (p, J_p) is some point of the space $Z^+(S^6, g_0)$, and $\mathcal{J} \in \{p, J_p\}$ is any section of the bundle, through the point. Then tangent space to $\mathcal{J}(S^6)$ is:

$$T_{(p, J_p)} \mathcal{J}(S^6) = \{(X, K) \in T_p S^6 \times T_{J_p} \pi^{-1}(p) : X = \frac{dx_t}{dt}|_{t=0}, K = \frac{dJ_{x_t}}{dt}|_{t=0}; x_t : I \longrightarrow S^6, x_0 = p\}$$

Lemma 3. For any point $(p, J_p) \in Z^+(S^6, g_0)$ any two sections \mathcal{J}^A and $\mathcal{J}^{\lambda A} \in \{p, J_p\}$ intersect transversely.

Proof. Tangent spaces $T_{(p, J_p)} \mathcal{J}^A(S^6)$ and $T_{(p, J_p)} \mathcal{J}^{\lambda}(S^6)$ are:

$$T_{(p, J_p)} \mathcal{J}^A(S^6) = \{(X, K) : X \in T_p S^6; K = A^{-1} R_{AX} A\}$$

$$T_{(p, J_p)} \mathcal{J}^{A\lambda}(S^6) = \{(X, K_\lambda) : X \in T_p S^6; K_\lambda = A^{-1} \lambda^{-1} R_{\lambda AX} \lambda A\}$$

$(X, K) = (X, K_\lambda)$ if and only if

$$K = K_\lambda$$

$$A^{-1} R_{AX} A(y) = A^{-1} \lambda^{-1} R_{\lambda AX} \lambda A(y)$$

for any vector $y \in T_p(S^6)$

$$R_{X'}(y') = \lambda^{-1} R_{\lambda X'} \lambda(y'),$$

where $X' = AX$, $X' \neq \pm p; 0$, $y' = A(y)$. This is possible just in case of $\lambda \in G_2$, but this contradicts to choice of λ .

Lemma 4. Any Cayley structure \mathcal{J}^A , passing through a point $(q, J_q) \in Z^+(S^6, g_0)$ is in family $\{q, J_q\}$.

Proof. Let Cayley structure \mathcal{J}^A comes through the point (q, J_q) , i.e. $\mathcal{J}_q^A = J_q$. Then by theorem 1 and lemma 1 $\mathcal{J}^A \in \{q, J_q\}$ just in case of existence the transformation $G \in G_2$, with $GA(q) = q$. Lie group G_2 acts transitively on sphere, so for any points $q, A(q) \in S^6$ one can find transformation $G \in G_2$, for which $GA(q) = q$.

Corollary 1. If any two Cayley structures are \mathcal{J}^A and \mathcal{J}^B intersect at a point $(q, J_q) \in Z^+(S^6, g_0)$, then $\mathcal{J}^A, \mathcal{J}^B \in \{q, J_q\}$.

3 Nearly Kähler $SU(3)$ -structures on S^6

We can see the above 1-parametric families using the differential forms point of view. Two forms (ω, ψ) on M^6 , where ω - is nondegenerate skewsymmetric 2-form, with stabilizer $Sp(3, \mathbb{R})$, and ψ - 3-form with stabilizer $SL(3, \mathbb{C})$ in each point, with some additional conditions define $SU(3)$ -structure M . Really, 3-form ψ gives reduction to $SL(3, \mathbb{C})$, and ω to $Sp(3, \mathbb{R})$. Group $SU(3)$ is intersection $Sp(3, \mathbb{R}) \cap SL(3, \mathbb{C})$, but to get $SU(3)$ -structure we need in two more algebraic conditions. To start remind that 3-form on M with $SL(3, \mathbb{C})$ as stabilizer defines almost complex structure J_ψ by formulas [7]:

$$K(X) = A(\iota_X \psi \wedge \psi)$$

where $A : \Lambda^5 \rightarrow TM \otimes \Lambda^6$ is isomorphism, induced by exterior product ($\iota_{A(\varphi)} Vol = \varphi$). Then, $\tau(\psi) = \frac{1}{6} \text{tr} K^2$, is section $\Lambda^6 \otimes \Lambda^6$ and $K^2 = Id \otimes \tau(\psi)$. It is known, that $SL(3, \mathbb{C})$ is stabilizer of 3-form ψ at each point, if and only if $\tau(\psi) < 0$. In this case, form ψ defines almost complex structure on M

$$J_\psi = \frac{1}{\kappa} K$$

where $\kappa = \sqrt{-\tau(\psi)}$.

The first of above algebraic properties is:

$$\omega \wedge \psi = 0$$

This is the property of ω to be of type (1,1) with respect to J . Second condition is positivity of form $\omega(X, JY) > 0$. In case, if the first and the second properties are hold, then forms (ω, ψ) define $SU(3)$ structure, and give almost complex structure J_ψ , metric $g(X, Y) = \omega(X, J_\psi Y)$, such that $g(J_\psi X, J_\psi Y) = g(X, Y)$. In this case almost Hermitian structure (g, J_ψ) is nearly Kähler if the following conditions are hold [8]:

$$\begin{cases} \psi = 3d\omega \\ d\phi = -2\mu\omega \wedge \omega \end{cases},$$

where $\iota_{JX}\phi = \iota_X\psi$.

Now, the cone of (M, g) is the Riemannian manifold $(\overline{M}, \overline{g})$ where $\overline{M} = M \times \mathbb{R}^+$ and $\overline{g} = r^2 g + dr^2$ in the coordinates (x, r) . Let define a section ρ of $\Lambda^3 \overline{M}$ by

$$\rho = r^2 dr \wedge \omega + r^3 \psi,$$

ρ is a generic 3-form, inducing a G_2 -structure on the 7-manifold \overline{M} such that \overline{g} is the metric determined by ρ , given the inclusion of G_2 in $SO(7)$. Moreover $\nabla^{\overline{g}} \rho = 0$, where $\nabla^{\overline{g}}$ is the Levi-Civita connection of \overline{g} . In other words, the holonomy of $(\overline{M}, \overline{g})$ is contained in G_2 .

Reciprocally, a parallel, generic 3-form on \overline{M} can always be written $\rho = r^2 dr \wedge \omega + r^3 \psi$, where (ω, ψ) define a nearly Kähler $SU(3)$ -structure on M .

The Riemannian cone of the 6-sphere is the Euclidean space \mathbb{R}^7 . According to what precedes, nearly Kähler structure on S^6 , compatible with g_0 , define a parallel or equivalently, a constant 3-form on \mathbb{R}^7 .

Let $x \in S^6$, and (ω_x, ψ_x) is some $SU(3)$ structure on $T_x S^6$. This structure is used to construct nearly Kähler structure on the sphere. Define constant 3-form on \mathbb{R}^7 :

$$\rho_{(x,1)} = dr \wedge \omega_x + \psi_x.$$

Then, ρ is parallel for the Levi-Civita connection of the flat metric and in return, $\omega = \iota_{\partial/\partial r}\rho$ and $\psi = \frac{1}{3}d\omega$ determine a nearly Kähler structure on S^6 whose values at x coincide with ω_x , ψ_x , consistent with our notations [4].

Let $\Psi_x = dz^1 \wedge dz^2 \wedge dz^3$ is complex volume form on $T_x S^6$, then $\psi_x = \text{Re}\Psi_x$ defines standard complex structure J_{0_x} on $T_x S^6 = \mathbb{R}^6$. Let $\lambda = \cos \varphi + i \sin \varphi$, $\varphi \in [0; \frac{2\pi}{3})$, consider family of forms $\Psi_{\lambda_x} = \lambda^3 \Psi_x$. Identify λ with unitary transformation $\lambda \cdot Id \in U(3)$, embedded into $SO(6)$ by standard way. Then $\psi_{\lambda_x}(X, Y, Z) = \psi_x(\lambda X, \lambda Y, \lambda Z)$ and $K_\lambda(\lambda X) = \lambda K(X)$, $J_{\lambda_x} = \lambda J_x \lambda^{-1} = J_x$. So, 1-parametric family of $SU(3)$ -structures $(\omega_x, \text{Re}(\lambda \Psi_x))$ defines family of nearly Kähler structures J_λ , equal to each other at point x .

Any another $SU(3)$ structure on $T_x S^6$ is defined by couple of forms (ω_A, ψ_A) , where $A \in SO(6)$, $\omega_A(X, Y) = \omega(A^{-1}X, A^{-1}Y)$, $\psi_A(X, Y, Z) = \psi(AX, AY, AZ)$. For those forms $K_A(AX) = AK(X)$ and $J_{A_x} = AJ_x A^{-1}$. So, at any $x \in S^6$ $SU(3)$ structures (ω_x, ψ_x) defines all possible almost complex structures on $T_x S^6$, for any J_x there exists 1-parametric family of nearly Kähler structures J_λ on S^6 , with $J_{\lambda_x} = J_x$.

Remark 3 *If p is some fixed point on the 6-sphere, then any $SU(3)$ structure on $T_x S^6$ defines nearly Kähler structure on S^6 . Set of those $SU(3)$ structures on 6-dimensional vector space is 7-dimensional homogeneous space $SO(6)/SU(3) \cong SO(7)/G_2 \cong \mathbb{R}P^7$. The fiber of twistor bundle is diffeomorphic to $SO(6)/U(3)$. Through "difference" between $U(3)$ and $SU(3)$ the above families of nearly Kähler structures arise.*

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